Consolidated Week 3 Notes: MAST30026 Aug 18, 2014

Lecture 1

*Def^{n}: Let (X,d) be a metric space. Let $x \in X$. A function $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ **converges** to x if $\vec{x}$ satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ & $n > N$ then $d(x_n, x) < \varepsilon$.

Note: Write $\lim_{n\to\infty} x_n = x$ if $\vec{x} = (x_1, x_2, \ldots)$ converges to x.

Note: A warning. $\lim_{y \to a} f(y)$ is a different kind of limit.

*Def^{n}: A function $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ **converges** to x if $\vec{x}$ satisfies:

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

* HW: Show that $\vec{x}: \mathbb{Z}_{>0} \to X$ satisfies the first definition of convergence if & only if $\lim_{n\to\infty} d(x_n, x) = 0$.

* HW: If $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ is a sequence on X & $x, y \in X$ & $\lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} x_n = y$ then $x = y$.

*Def^{n}: Let $d_1: X \times X \to \mathbb{R}_{>0}$ & $d_2: X \times X \to \mathbb{R}_{>0}$ be metrics on X. The metrics d_1 & d_2 are **equivalent** if d_1 & d_2 satisfy:

if $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ & $x \in X$ then $\lim_{n\to\infty} d_1(x_n, x) = 0$ if & only if $\lim_{n\to\infty} d_2(x_n, x) = 0$.

* HW: Show that if d_1 & d_2 satisfy:

if $x, y \in X$ then there exists $c_1 \in \mathbb{R}_{>0}$ & $c_2 \in \mathbb{R}_{>0}$ such that

$d_1(x,y) \leq c_1 d_2(x,y)$ & $d_2(x,y) \leq c_2 d_1(x,y)$

then d_1 & d_2 are equivalent metrics.

* HW: Does lemma 2.12 in the notes use the above definition of equivalent metrics OR does it use the following:

There exist $c_1 \in \mathbb{R}_{>0}$ & $c_2 \in \mathbb{R}_{>0}$ such that

if $x, y \in X$ then $d_1(x,y) \leq c_1 d_2(x,y)$ & $d_2(x,y) \leq c_2 d_1(x,y)$

then d_1 & d_2 are equivalent metrics?

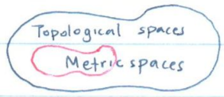
* HW: Why aren't the above definitions of equivalent metrics "if & only if" statements?

** Theorem: Let (X,d) be a metric space. X is a topological space with the metric space topology (the open sets are unions of open balls; an open ball is $B(x,\varepsilon) = \{y \in X \mid d(y,x) < \varepsilon\}$). Let $A \subseteq X$ & let $\overline{A}$ be the closure of A (the smallest closed set containing A).

Then $\overline{A} = \{ x \in X \mid \text{there exists } \vec{a}: \begin{matrix} \mathbb{Z}_{>0} \to A \\ n \mapsto a_n \end{matrix} \text{ with } \lim_{n\to\infty} a_n = x \}$.

Lecture 2

*Def$^{\text{n}}$: Let X be a topological space. Let $A \subseteq X$. The **boundary** of A is
$$\partial A = \overline{A} \cap \overline{(A^c)}$$



Note: $\overline{A} = A \cup \partial A$

*Def$^{\text{n}}$: The set A is **dense** in X if $\overline{A} = X$.

*Def$^{\text{n}}$: The set A is **nowhere dense** in X if $(\overline{A})^\circ = \emptyset$

– Examples: (1) $\mathbb{Q}$ is dense in $\mathbb{R}$

(2) $(0,1]$ is dense in $[0,1]$

(3) The boundary of $\mathbb{Q}$ in $\mathbb{R}$ is $\overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^c} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$

(4) The boundary of $(0,1]$ in $\mathbb{R}$ is
$$\begin{aligned} \partial (0,1] &= \overline{(0,1]} \cap \overline{(0,1]^c} \\ &= [0,1] \cap \overline{((-\infty,0] \cup (1,\infty))} \\ &= [0,1] \cap ((-\infty,0] \cap [1,\infty)) \\ &= \{0,1\} \end{aligned}$$

(5) $\mathbb{Z}_{>0}$ & $\mathbb{Z}$ are nowhere dense

(6) $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$

(7) The Cantor set is nowhere dense in $[0,1]$

*Def$^{\text{n}}$: Let (X,d) be a metric space. Let $A \subseteq X$. The set A is **bounded** if A satisfies:

there exists $M \in \mathbb{R}_{>0}$ such that

if $a_1, a_2 \in A$ then $d(a_1,a_2) \leq M$.

– Example: $\mathbb{Z}_{>0}$ is not bounded in $\mathbb{R}$.

* HW: Let (X,d) be a metric space. Let $\vec{x}: \begin{matrix} \mathbb{Z}_{>0} \to X \\ n \mapsto x_n \end{matrix}$ be a sequence in X. Show that if $\vec{x}$ converges then $\{x_1, x_2, x_3, \dots\}$ is bounded.

*Def$^{\text{n}}$: Let (X,d) & (C,ρ) be metric spaces. Let $f: X \to C$. The function f is **continuous** if f satisfies:

if $x \in X$ & $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in X$ & $d(x,y) < \delta$ then $\rho(f(x),f(y)) < \varepsilon$.

The function f is **uniformly continuous** if f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x \in X$ & $y \in X$ & $d(x,y) < \delta$ then $\rho(f(x),f(y)) < \varepsilon$

* HW: Let $f: X \to Y$ & $g: Y \to Z$ be continuous. Show that $g \circ f$ is continuous.

* HW: Let $A \subseteq X$. Let $f: X \to Y$ be continuous. Show that $g: \begin{matrix} A \to Y \\ a \mapsto f(a) \end{matrix}$ is continuous.

* HW: Let $f_1 : X_1 \to Y_1$ & $f_2 : X_2 \to Y_2$ be continuous. Show that $f : X_1 \times X_2 \to Y_1 \times Y_2$, $(x_1, x_2) \mapsto (f(x_1), f(x_2))$ is continuous.

* HW: Show that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x,y) \mapsto x+y$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x,y) \mapsto xy$ and $h : \mathbb{R} \to \mathbb{R}$, $x \mapsto -x$ are continuous.

Lecture 3

1. *Def^n: Let (X, d) & (C, ρ) be metric spaces & $f : X \to C$ be a function. The function $f : X \to C$ is continuous if f satisfies:

   if $x \in X$ & $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

   if $y \in X$ & $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \varepsilon$.

2. *Def^n: Let $(X, \mathcal{T})$ & $(C, \mathcal{R})$ be topological spaces. Let $f : X \to C$ be a function. The function $f : X \to C$ is continuous if f satisfies:

   if V is open in C ($V \in \mathcal{R}$) then $f^{-1}(V)$ is open in X ($f^{-1}(V) \in \mathcal{T}$)

   Note: Continuity is really about topological spaces. (Continuous functions are the morphisms in the category of topological spaces).

* HW: Show that if $f : X \to C$ satisfies 1. if & only if $f : X \to C$ satisfies 2. where (X, d) & (C, ρ) are viewed as topological spaces with the metric space topology.

3. *Def^n: Let (X, d) & (C, ρ) be metric spaces & $f : X \to C$ a function. The function $f : X \to C$ is continuous if f satisfies:

   if $x \in X$ then $\lim_{y \to x} f(y) = f(x)$.

   *Def^n: Let $z \in C$. The limit of f as y approaches x is z if $f : X \to C$ satisfies:

   if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

   if $y \in X$ & $d(y, x) < \delta$ then $\rho(f(y), z) < \varepsilon$

4. *Def^n: Let (X, d) & (C, ρ) be metric spaces & $f : X \to C$ a function. The function $f : X \to C$ is continuous if f satisfies:

   if $x \in X$ & $\vec{x} : \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ with $\lim_{n \to \infty} x_n = x$ then $\lim_{n \to \infty} f(x_n) = f(x)$.

   *Def^n: The function $f : X \to C$ is uniformly continuous if f satisfies

   if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

   if $x, y \in X$ & $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \varepsilon$.

* HW: If $f : X \to C$ is uniformly continuous, show then that f is continuous.

* HW: Give an example of a function which is continuous, but not uniformly continuous.

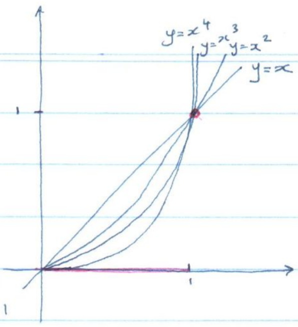
* HW: Show that $f : \mathbb{R} \to \mathbb{R}_{>0}$, $f(x) = \frac{1}{1+x^2}$ is uniformly continuous.

* HW: Show that $f : \mathbb{R} \to \mathbb{R}_{>0}$, $f(x) = x^2$ is continuous, but not uniformly continuous.

- Example: $f : \mathbb{R} \setminus \{0\} \to [1,1]$, $f(x) = \sin(\frac{1}{x})$ is not uniformly continuous.

- Examples: (1) $f_n : [0,1) \longrightarrow [0,1)$ for $n \in \mathbb{Z}_{>0}$
$x \longmapsto x^n$

Then $\lim_{n\to\infty} f_n = f$ where $f : [0,1) \longrightarrow [0,1)$
$x \longmapsto 0$



(2) $f_n : [0,1] \longrightarrow [0,1]$ for $n \in \mathbb{Z}_{>0}$
$x \longmapsto x^n$

Then $\lim_{n\to\infty} f_n = f$ where $f : [0,1] \longrightarrow [0,1]$
$$x \longmapsto \begin{cases} 0, \text{ if } x < 1 \\ 1, \text{ if } x = 1 \end{cases}$$

(3) $f_n : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ for $n \in \mathbb{Z}_{>0}$
$x \longmapsto x^n$

Then $\lim_{n\to\infty} f_n$ does not exist.

*Def^n: Let (X,d) & (C,ρ) be metric spaces. Let $F = \{f : X \to C\}$

Define: $\sigma : F \times F \longrightarrow \mathbb{R}_{\geq 0}$ by $\sigma(f,g) = \sup\{d(f(x),g(x)) \mid x \in X\}$
$(f,g) \longmapsto \sigma(f,g)$

Note: σ is not quite a metric.

Let $f_1, f_2, \dots$ be a sequence of functions from X to C in F.

Let $f : X \to C$ be a function

The sequence $f_1, f_2, \dots$ converges pointwise to f if $f_1, f_2, \dots$ satisfies:
if $x \in X$ then $\lim_{n\to\infty} d(f_n(x), f(x)) = 0$.

The sequence $f_1, f_2, \dots$ converges uniformly to f if $f_1, f_2, \dots$ satisfies:
$\lim_{n\to\infty} \sigma(f_n, f) = 0$.

End of Week 3.